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AN OUTLINE OF THE TRANSITION PROBABILITY FUNCTION APPROACH TO S--ETC(U)
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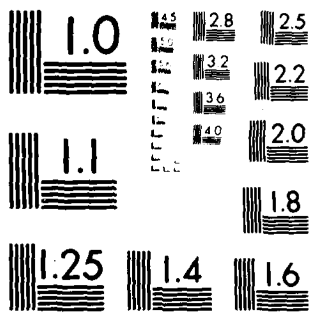

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AN OUTLINE OF THE TRANSITION PROBABILITY FUNCTION APPROACH TO STOCHASTIC SYSTEMS

JAMES A. RENEKE

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NAVAL COASTAL SYSTEMS CENTER
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This report was written by James A. Reneke, Clemson University, South Carolina, while at NCSC under the 1981 Navy-ASEE Summer Faculty Research Program sponsored by ONR. Work was related to Code 795's research in the areas of parameter estimation, control system design, and nonlinear stochastic system analysis. The work was performed in the period May to August 1981. Further information may be obtained from Gerald Dobeck, Code 795, NCSC.

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Problems of estimation, control, and qualitative analysis for noisy engineering systems can be approached either from the sample path or the transition probability function perspective. The second approach is outlined in this report in a series of examples with enough theory to make the material self contained. Special attention is given the relationship between engineering models and Kolmogorov's partial differential equations.

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SECTION I

INTRODUCTION

OBJECTIVES

Problems of estimation, control, and qualitative analysis for noisy engineering systems are central to the Control System Test Vehicle (CSTV) project. The work was concerned with an approach to these problems via the transition probability perspective which has promise for some problems which fall outside the domain of the linear-quadratic-Gaussian techniques. In particular, the methods are applicable to the nonlinear models of the CSTV.

The relevant part of the literature on Markov processes is unusually profound, requiring knowledge of probability theory and dynamical systems. This report attempts to explain that part of the theory which is applicable to the CSTV models by considering a series of examples and special cases. For instance, the general theory treats the model

$$dX(t) = b[X(t)]\, dt + \sigma[X(t)]\, dW(t)$$

with the Ito integral as opposed to the Riemann integral of ordinary calculus because $\sigma(x)$ depends on the state.[1] The coefficient $\sigma(x)$ is constant for the CSTV models affording a great simplification. We will only treat this special case.

The transition probability approach is relevant in that it allows for refinements in current techniques; i.e., in estimating the steady-state distribution of the state vector, in nonlinear parameter estimation, and in control. These refinements are important because the nonlinear models might exhibit complex behavior such as multiple modes with a resultant quasi-periodic motion which cannot be dealt with using the LQG techniques.

The material in this report is presented informally in terms of examples with the intention of illustrating the nature of the problems and techniques of solution; we are trying to present an outline or road map rather than a detailed survey.

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[1]Gelb, A. (Ed), Applied Optimal Estimation, The M. I. T. Press, 1974.

ORGANIZATION

Section II begins with a discussion of the general mathematical background. Since we will consider a part of the general theory of Markov processes, we will attempt to define the distinctions between the two approaches to Markov processes; i.e., the sample path approach to which engineering models most naturally belong and the transition probability approach.

The partial differential equations of Kolmogorov which are of particular mathematical importance in this report are discussed in Section III. A brief derivation of the equations and a discussion of the relationship between the partial differential equations and stochastic differential equations (the engineering models) are included. The problems of existence of solutions (well posedness) and asymptotic results are mentioned.

The problems of parameter estimation, the most developed application of the methods outlined in Section III, are discussed in Section IV. Section V discusses the qualitative analysis of noisy system and Section VI the control of noisy systems.

SECTION II

BACKGROUND

TRANSITION PROBABILITY FUNCTIONS

In systems theory, the study of a system most often begins with a model, typically $\dot{X} = AX + BU$. In the theory of stochastic processes, some of which have models of the form $dX(t) = b[X(t)]\,dt + \sigma\,dW(t)$, the study is much more likely to begin with functions defined on the sample paths $\{X(t), t \geq 0\}$. For instance, the system concept of state (the Markov property: given the present, the future is independent of the past[2]) is defined in terms of conditional distributions on $\{X(t), t \geq 0\}$ rather than as a property of some model. Markov processes are usually described in the literature[3 4 5] by transition probability functions.

[2]Parzen, E., Stochastic Processes, Holden-Day, 1962.

[3]Dynkin, Y. B., Markov Process, Vols. I and II, Springer-Verlag, 1965.

[4]Fuller, W., An Introduction to Probability Theory and Its Applications, Vol. I, Third Edition, John Wiley & Sons, 1968.

[5]Gikhman, I. I. and Skorohod, A. V., The Theory of Stochastic Processes, Vol. I through III, Springer-Verlag, 1974, 1975, 1979.

A function $P(E,t|x,t_0)$ is a transition probability function on a Markov process $\{X(t),t\geq 0\}$ provided $P(E,t|x,t_0)$ is the conditional probability that $X(t)$ belongs to E, given that at time $t_0 < t$ we have $X(t_0) = x$. The process is stationary or homogeneous in time if $P(E,t|x, t_0) = P(E,t-t_0|x,0)$. We are only concerned with this case and write $P(E,t|x)$ in the place of $P(E,t|x,0)$.

The transition probability function satisfies the Chapman-Kolmogorov equation[6]

$$P(E,t + s|x) = \int_{R^n} P(E,t|y)\, P(dy, s|x).$$

If a transition density $p(y,t|x)$ exists, i.e., if $P(E,t|x) = \int_E p(y,t|x)dy$, then

$$p(y,t + s|x) = \int_{R^n} p(y,t|z)\, p(z,s|x)\, dz.$$

Note that if $p(\cdot,t|x)$ is known for $0 \leq t \leq \varepsilon$, then $p(\cdot,t|x)$ can be found for all t by iterating. Under suitable continuity conditions and with proper interpretations $p(\cdot,0|x)$ and $\partial/\partial t(p(\cdot,t|x))|_{t=0}$ completely determine $p(\cdot,t|x)$ for all t.[7] Furthermore, $p(\cdot,0|x)$ is always the same; viz, $p(y,0|x) = \delta(y-x)$.

There is extensive literature on Markov processes dealing with both the mathematical foundations and applications. The literature can be considered to be composed of three parts.[2]

1. The study of time dependent behavior: to find the transition probability function by finding and solving (differential, integral, or other functional) equations which is satisfies.

2. The study of long run behavior: to find conditions under which a steady state exists; i.e., $\lim_{t\to\infty} P(E,t|x) = \pi(E)$.

3. The study of the qualitative behavior: to examine the probability distribution of the amount of time the system spends in various states

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[2]ibid.

[6]Arnold, L., Stochastic Differential Equations: Theory and Applications, Interscience, 1974.

[7]Wong, L., Stochastic Processes in Information and Dynamic Systems, McGraw-Hill, 1971.

and the length of time required for the system to pass from one set of states to another.

The first two areas will be discussed in Section III and the third area in Section V.

The transition probability function approach to engineering problems can be justified in three ways.

1. This approach provides a distinctive modeling technique.

2. The models are deterministic dynamical systems which allow for application of classical methods.

3. This approach leads to more sophisticated models than the sample path approach.

SAMPLE PATH VERSUS TRANSITION PROBABILITIES

One approach to modeling engineering systems operating in noisy environments or systems for which only noisy measurements can be made is to modify deterministic models, i.e., differential equation models assuming complete knowledge, by replacing parameters with stochastic processes to obtain *stochastic differential* equations.[8] There is a lot of flexibility in this approch even if it is restricted to the Wiener process (the "derivative" of a Wiener process is white noise) or to processes obtained from a Wiener process. There is some controversy in the literature on the appropriateness of the resulting models,[8] [9] but the issues of the controversy do not seem to apply to the CSTV models. For our models, the noise terms are state independent.

One approach for obtaining transition probability models begins with a discrete model and produces the final model through a limit. The canonical example of this approach is the diffusion equation model of Brownian motion obtained from a random walk model.[10] Another example is that of a population model obtained from a discrete birth/death process model.[11] There is a feeling in this approach that the discrete model is "microscopic" and the continuous model obtained as a limit is "macroscopic." We will see in Section III

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[8]McShane, E. J., Stochastic Calculus and Stochastic Models, Academic Press, 1974.

[9]Gray, A. H. and Caughey, J. K., A Controversy in Problems Involving Random Parametric Excitation, J. Math and Phys., 44 (1975), pp. 288-296.

[10]Wax, N., Selected Papers on Noise and Stochastic Processes, Dover Publications, 1954.

[11]Bartlett, M. S., Stochastic Population Models in Ecology and Epidemiology, Methune, London, 1960.

how to relate the stochastic differential equation models with the transition probability models.

The stochastic differential equation models are often nonlinear. Although we have satisfactory computational methods for models which are linear, methods for the nonlinear models are much less developed.[1] Often our only option in analyzing these nonlinear models is simulation. While simulation is a powerful technique, it is cumbersome. On the other hand, the transition probability models are deterministic dynamical systems. Frequently, the resulting model is a parabolic or elliptic partial differential equation of a type that has been studied for decades and for which numerical methods are highly developed.[12]

The transition probability approach requires a lot of mathematical sophistication. Many deep results from analysis, probability theory, and dynamical systems are called on. However, the resulting models and analysis are both very elegant and powerful.

Finally, the literature seems to be divided into two camps: those identified with the sample path approach and those identified with the transition probability approach. Since both approaches are attacks on the same problem, the analysis of noisy systems, we should be able to make use of both.

SECTION III

THE PARTIAL DIFFERENTIAL EQUATIONS OF KOLMOGOROV

RELATING THE TWO APPROACHES

Suppose that $dX(t) = b[X(t)]\,dt + \sigma\,dW(t)$; i.e.,

$$X(t+\Delta t) - X(t) = \int_t^{t+\Delta t} b[X(s)]\,ds + \sigma\,[W(t+\Delta t) - W(t)]$$

$$= b[X(\bar{t})]\,\Delta t + \sigma\,[W(t+\Delta t) - W(t)].$$

Then

$$\frac{E[X(t+\Delta t) - X(t)]}{\Delta t} = E[b(X(\bar{t})]$$

$$= \int_{-\infty}^{\infty} b(y)\,p(y,\bar{t}\,|x)dy,$$

[12]Bers, L., John, F., and Schechter, M., Lectures in Applied Mathematics, Vol. III, Partial Differential Equations, Interscience, 1964.

where $\bar{t}\varepsilon(t, t + \Delta t)$ and $X(t) = x$. Hence,

$$\frac{E\,[X(t + \Delta t) - X(t)]}{\Delta t} \rightarrow b(x) \text{ as } \Delta t \rightarrow 0.$$

Also,

$$E([X(t + \Delta t) - X(t)]\,[X(t + \Delta t) - X(t)]^T)$$

$$= E(b[X(\bar{t})]\,b[X(\bar{t})]^T)\,(\Delta t)^2 + E(b(X(\bar{t}))(W(t + \Delta t) - W(t)))\,\sigma\Delta t$$

$$+ \sigma^2\,E([W(t + \Delta t) - W(t)]\,[W(t + \Delta t) - W(t)]^T).$$

Hence

$$\frac{E([X(t + \Delta t) - X(t)]\,[X(t + \Delta t) - X(t)]^T)}{\Delta t} \rightarrow \sigma^2 \text{ as } \Delta t \rightarrow 0.$$

Note that

$$\frac{E(X(t + \Delta t) - X(t))}{\Delta t} = \frac{1}{\Delta t} \int_{-\infty}^{\infty} (y - x)\, p(y,\Delta t|x)dy$$

and

$$\frac{E([X(t + \Delta t) - X(t)][X(t + \Delta t) - X(t)]^T)}{\Delta t}$$

$$= \frac{1}{\Delta t} \int_{-\infty}^{\infty} (y - x)(y - x)^T p(y,\Delta t|x)\, dy.$$

This computation leads to the following definition.[13] A Markov process $\{X(t), t \geq 0\}$ with continuous sample paths is called a diffusion process[14] if its transition density $p(y,t|x)$ satisfies for $\varepsilon>0$ as $t\rightarrow0$

1. $\frac{1}{t} \int_{|y-x|>\varepsilon} p(y,t|x)\, dy\rightarrow0$

2. $\frac{1}{t} \int_{|y-x|\leq\varepsilon} (y - x)p(y,t|x)\, dy \rightarrow b(x)$ (drift)

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[13]Kolmogorov, A., Uber die Analytische Methoden in der Wahrscheinlichkeitsrechnung, Math. Ann. 104, pp. 415-458, 1931.

[14]Stroock, D. and Varadhau, S. R. S., Diffusion Processes with Boundary Conditions, Comm. Pure Appl. Math., 24(1971), pp. 147-225.

and

$$3. \quad \frac{1}{t} \int_{|y-x| \leq \varepsilon} (y - x)(y - x)^T \, p(y,t|x) \, dy \to a(x) \qquad \text{(dispersion)}.$$

We are concerned with diffusions for which the dispersion coefficient is not state dependent; i.e., $a(x)$ is constant. However, for the moment let us consider the general one-dimensional case.

For sufficiently nice functions $f(x)$, we have

$$u(t,x) = \int_{-\infty}^{\infty} f(y) \, p(y,t|x) \, dy$$

which is defined for all $t \geq 0$ and $-\infty < x < \infty$. We want to show that u solves the backward equation[13]

$$\frac{\partial u}{\partial t} = \frac{1}{2} a(x) \frac{\partial^2 u}{\partial x^2} + b(x) \frac{\partial u}{\partial x}$$

$$u(0,x) = f(x).$$

Recalling[15] the Chapman-Kolmogorov equation

$$p(y,t + \Delta t|x) = \int_{-\infty}^{\infty} p(y,t|z) \, p(z,\Delta t|x) \, dz ,$$

we have

$$u(t + \Delta t,x) - u(t,x) = \int_{-\infty}^{\infty} f(y) \left\{ \int_{-\infty}^{\infty} p(y,t|z)p(z,\Delta t \; x) \, dz - p(y,t|x)\right\} dy$$

$$= \int_{-\infty}^{\infty} f(y) \left\{ \int_{-\infty}^{\infty} \left\{p(y,t|x) + \frac{\partial p(y,t|x)}{\partial x} (z - x) \right.\right.$$

$$\left.\left. + \frac{1}{2} \frac{\partial^2 p(y,t \; x)}{\partial x^2} (z - x)^2 + 0(|z - x|^2)\right\} p(z,\Delta t|x) \, dz - p(y,t|x)\right\} dy$$

---

[13]ibid.

[15]Brannan, J., Classroom Notes, Department of Mathematical Sciences, Clemson University, 1980-81.

$$= \int_{-\infty}^{\infty} f(y)\{\int_{-\infty}^{\infty} (z - x)p(z,\Delta t|x)\ dz\ \frac{\partial p(y,t|x)}{\partial x}$$

$$+ \int_{-\infty}^{\infty} (z - x)^2\ p\ (z,\Delta t|x)\ dy \cdot \tfrac{1}{2} \frac{\partial^2 p(y,t|x)}{\partial x^2}$$

$$+ \int_{-\infty}^{\infty} 0(|z - x|^2)\ p(z,\Delta t|x)\ dz\}\ dy.$$

Hence

$$\frac{u(t + \Delta t,x) - u(t,x)}{\Delta t} \rightarrow \int_{-\infty}^{\infty} f(y)\{b(x)\ \frac{\partial p(y,t|x)}{\partial x} + \tfrac{1}{2}\ a(x)\ \frac{\partial^2 p(y,t|x)}{\partial x^2}\}\ dy$$

as $\Delta t \rightarrow 0$.

On the other hand,

$$\frac{\partial u(t,x)}{\partial x} = \int_{-\infty}^{\infty} f(y)\ \frac{\partial p(y,t|x)}{\partial x}\ dy$$

and

$$\frac{\partial^2 u(t,x)}{\partial x^2} = \int_{-\infty}^{\infty} f(y)\ \frac{\partial p(y,t|x)}{\partial x}\ dx.$$

Therefore

$$\frac{\partial u}{\partial t}\ (t,x) = b(x)\ \frac{\partial u}{\partial x}\ (t,x) + \tfrac{1}{2}a(x)\ \frac{\partial^2 u}{\partial x^2}\ (t,x).$$

Since $p(y,0|x) = \delta(y - x)$ we also have $u(0,x) = f(x)$. The name "backward equation" evidently comes from the fact that y and t are held constant and we are looking backward in time to differentiate $p(y,t|x)$ with respect to x.[15]

The general backward equation of Kolmogorov is

$$\frac{\partial u}{\partial t} = \tfrac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x)\ \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x)\ \frac{\partial u}{\partial x_j}$$

$$u(0,x) = f(x).$$

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[15]ibid.

The probabilistic interpretation of u is that u(t,x) is the expected value of f(X(t)) given that X(0) = x.

We proceed now with the forward equation.[15] For the backward equation the n-dimensional case is essentially the same as the one-dimensional case. For the forward equation the n-dimensional case is essentially the same as the two-dimensional case. So we consider n = 2.

For a sufficiently nice function $u_0(z) = u_0(z_1,z_2)$ from R^2 to R^1 the Chapman-Kolmogorov equation yields

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_0(z)\, p(z,t+\tau|x)\, dz_1 dz_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_0(z)\, p(z,t|y)\, p(y,\tau|x)\, dy_1 dy_2 dz_1 dz_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(t,y)\, p(y,\tau|x)\, dy_1 dy_2,$$

where

$$u(t,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_0(z)\, p(z,t|y)\, dz_1 dz_2.$$

Further,

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_0(z)\, p(z,t+\tau|x)\, dz_1 dz_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_0(z)\, \frac{\partial}{\partial t} p(z,t+\tau|x)\, dz_1 dz_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_0(z)\, \frac{\partial}{\partial t} p(z,t+\tau|x)\, dz_1 dz_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial}{\partial t} u(t,y)\, p(y,\tau|x)\, dy_1 dy_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{\frac{1}{2}\sum_{i,j=1}^{2} a_{ij}(y)\, \frac{\partial^2 u(t,y)}{\partial y_i \partial y_j} + \sum_{j=1}^{2} b_j(y)\, \frac{\partial u(t,y)}{\partial y_j}\right\} p(y,\tau|x)\, dy_1 dy_2$$

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[15] ibid.

We next apply integration by parts which for each i and j yields

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a_{ij}(y)\,\frac{\partial^2 u(t,y)}{\partial y_i \partial y_j}\,p(y,\tau|x)\;dy_1 dy_2$$

$$= \int_{-\infty}^{\infty} \{a_{ij}(y)\,\frac{\partial u(t,y)}{\partial y_j}\;p(y,\tau|x)\;\Big|_{y_i=-\infty}^{y_i=\infty}$$

$$- \int_{-\infty}^{\infty} \frac{\partial u(y,t)}{\partial y_j}\,\frac{\partial}{\partial y_i}\,(a_{ij}(y)p(y,\tau|x)dy_i\}\;dy_j$$

$$= - \int_{-\infty}^{\infty} \{u(y,t)\,\frac{\partial}{\partial y_i}\,(a_{ij}(y)p(y,\tau|x))\;\Big|_{y_j=-\infty}^{y_j=\infty}$$

$$- \int_{-\infty}^{\infty} u(y,t)\,\frac{\partial^2}{\partial y_i y_j}\,(a_{ij}(y)p(y,\tau|x)dy_j\}\;dy_i$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(y,t)\,\frac{\partial^2}{\partial y_i \partial y_j}\,(a_{ij}(y)p(y,\tau|x))\;dy_1 dy_2 \quad .$$

A similar process yields for each j

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} b_j(y)\,\frac{\partial}{\partial y_j}\,u(y,t)p(y,\tau|x)\;dy_1 dy_2$$

$$= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial}{\partial y_j}\,(b_j(y)p(y,\tau|x))\;dy_1 dy_2 \quad .$$

So we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_0(y)\,\frac{\partial}{\partial t}\,p(y,t+\tau|x)\;dy_1 dy_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(t,y)\,\{\tfrac{1}{2}\sum_{i,j=1}^{2}\;\frac{\partial^2}{\partial y_i \partial y_j}\,(a_{ij}(y)p(y,\tau|x))$$

$$- \sum_{j=1}^{2}\,\frac{\partial}{\partial y_j}\,(b_j(y)p(y,\tau|x))\}\;dy_1 dy_2 \quad .$$

As $t \to 0$, $u(t,y) \to u_0(y)$. Hence

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_0(y)\{\frac{\partial}{\partial \tau} p(y,\tau|x) - \tfrac{1}{2} \sum_{i,j=1}^{2} (a_{ij}(y)p(y,\tau|x))$$

$$+ \sum_{j=1}^{2} \frac{\partial}{\partial y_j} (b_j(y)p(y,\tau|x))\} \, dy_1 dy_2 = 0$$

for all sufficiently nice $u_0(y)$; i.e., the density must satisfy the forward equation of Kolmogorov[13]

$$\frac{\partial}{\partial t} p(y,t|x) = \tfrac{1}{2} \sum_{i,j=1}^{2} \frac{\partial^2}{\partial y_i \partial y_j} (a_{ij}(y)p(y,t|x))$$

$$- \sum_{j=1}^{2} \frac{\partial}{\partial y_j} (b_j(y)p(y,t|x))$$

$$p(y,0\ x) = \delta(y - x).$$

WELL POSED PROBLEMS

For both the forward and backward equations we have only imposed initial conditions which, in general, are not sufficient to assure a unique solution.[12] We must introduce either boundary conditions or some other condition in the absence of a boundary to match up a solution of the PDEs with the stochastic differential equation. This report treats in detail only examples from the no-boundary case.

The best introduction to the problem of appropriate boundary conditions is Feller's paper on one-dimensional diffusions.[14] [16] The scope of the problems is discussed here.

Consider the problem of modeling a constrained pendulum; i.e., a simple pendulum with a tagline limiting the pendulum's arc. On the free portion of the arc the system might be modeled by

$$\ddot{x} + a\dot{x} + bx = f(t)$$

$$x > x_0,$$

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[12]ibid.

[13]ibid.

[14]ibid.

[16]Feller, W., Diffusion Processes in One Dimension, Trans. Amer. Math. Soc., 77(1954), pp. 1-31.

where f(t) represents an exogenous disturbance. In vector form the model becomes

$$\begin{cases} \dot{X}(t) = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} X(t) + \begin{bmatrix} 0 \\ f(t) \end{bmatrix} \\ x_1(t) > x_o \end{cases} .$$

Modeling the disturbance as white noise the stochastic differential equation becomes

$$dX(t) = BX(t)\ dt + \sigma\ dW(t)$$

where

$$B = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \text{ and } \sigma = \begin{bmatrix} 0 & 0 \\ 0 & \varepsilon \end{bmatrix}.$$

The backward equation is

$$\frac{\partial u}{\partial t} = \tfrac{1}{2}\, \varepsilon^2 \frac{\partial^2 u}{\partial x_2^2} + x_1 \frac{\partial u}{\partial x_1} - (bx_1 + ax_2) \frac{\partial u}{\partial x_2} .$$

The model is obviously incomplete since we have not specified what happens on the boundary $x_1 = x_o$. If we want to model the process as a diffusion, then the boundary must be modeled as an absorbing, reflecting, or elastic barrier, or the process at the boundary must be modeled as an elementary return process.[16] We stop here, but the point to be made is that the theory includes a very rich class of models.

We are getting very deep into the theory of dynamical systems; however, we must continue a bit further to get a feeling for how all of the pieces fit together. Recall the relation

$$u(t,x) = \int_{-\infty}^{\infty} f(y)p(y,t|x)\ dy.$$

Since u depends on f, t, and x, let us rewrite the relation as

$$[T_t f](x) = \int_{-\infty}^{\infty} f(y)p(y,t|x)\ dy .$$

Thus for each function f we can think of $T_t f$ as a function; i.e., T_t is a function from some set of functions to a set of functions. Let us call this

[16] ibid.

set G. So far each t, T_t is a function from G into G and $\{T_t, t \geq 0\}$ is called a semigroup.[17]

The study of semigroups has a large literature in its own right. The modern theory of Markov processes is largely semigroup theory.[7] Of interest to us is that the study of the forward and backward questions is subsumed in semigroups and the problem of appropriate boundary conditions or other conditions specifying the solution for the PDEs when the boundaries are absent is translated into conditions on the space G.[12]

ASYMPTOTIC RESULTS

The qualitative analysis of dynamical systems is largely the description of the long term behavior of the system. For deterministic systems, this involves the determination of the stable points of the system. The behavior of stochastic systems is much richer; instead of stable points, we look for steady-state distributions or densities.[18] Since we usually deal with systems whose state vector takes on every permissible value infinitely often, we cannot talk of stable points but only of the relative amount of time the state vector is nearby some fixed point.

Obtaining asymptotic results for the backward equation seems to be easier than for the forward equation. They can then be translated into results for the forward equation. Furthermore, we seem to have a better idea of what to look for; i.e., we want $u(t,x)$ to approach a constant function as $t \to \infty$. Therefore, the expected value of $f(X(t))$, as $t \to \infty$, is independent of the starting value of $X(t)$, a reasonable way for physical systems to behave.

We only have partial results to report in this area, so treatment is left a little skimpy.[19] However, the results do show in outline how semigroup theory is used.

Consider the one-dimensional problem

$$\frac{\partial u}{\partial t} = \tfrac{1}{2}\, a(x)\, \frac{\partial^2 u}{\partial x^2} + b(x)\, \frac{\partial u}{\partial x}$$

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[7] ibid.

[12] ibid.

[17] Curtain, R. F. and Pritchard, A. J., Infinite Dimensional Linear Systems Theory, Springer-Verlag, 1978.

[18] Papanicolaou, G. C. and Kohler, W., Asymptotic Analysis of Deterministic and Stochastic Equations with Rapidly Varying Components, Comm. Math. Phys., 45 (1975), pp. 217-232.

[19] Brannan, J. and Reneke, J., Unpublished Research, 1981.

on $-\infty < x < \infty$. We seek conditions on $a(x)$ and $b(x)$ which ensure that $u(t,x)$ has a constant limit as $t \to \infty$.

One such set of conditions is:

1. $a(x)$ and $b(x)$ are continuous on $(-\infty,\infty)$ and $a(x) > 0$ for all x.

2. The function $f(x)$ defined by

$$f(x) = \exp\left(2 \int_0^x b(s)/a(s)\, ds\right)/a(x)$$

is absolutely summable on $(-\infty, \infty)$; i.e.,

$$\int_{-\infty}^{\infty} |f(x)|dx$$

exists and is finite. For example, if $dX(t) = -X^3(t)\, dt + \varepsilon\, dW(t)$, then the expected value of $X(t)$ as $t \to \infty$ is independent of $X(0)$.

Let k and m denote the increasing functions defined on $(-\infty, \infty)$, respectively, by

$$k(t) = \int_0^t \exp\left(-2 \int_0^x b(s)/a(s)\, ds\right) dx$$

and

$$m(t) = \int_0^t 2 \exp\left(2 \int_0^x b(x)/a(s)\, ds\right)/a(x)\, dx.$$

Let G denote the complex valued functions f on $(-\infty, \infty)$ such that each of

$$\int_{-\infty}^{\infty} |f|^2\, dm = \int_{-\infty}^{\infty} |f(x)|^2\, m'(x)\, dx$$

and

$$\int_{-\infty}^{\infty} |df|^2/dk = \int_{-\infty}^{\infty} |f'(x)|^2/k'(x)\, dx$$

exists and is finite. Let $\langle\cdot,\cdot\rangle$ denote the inner product defined on G by

$$\langle f,h\rangle = \int_{-\infty}^{\infty} f\bar{h}\, dm + \int_{-\infty}^{\infty} df\, d\bar{h}/dk.$$

A consequence of the Hille-Yoshida Theorem[17] is that we can define a semigroup $\{T_t,\ t \geq 0\}$ on G by $[T_t f](x) = u(t,x)$ provided

$$\frac{\partial u}{\partial t} = \tfrac{1}{2}\, a(x)\, \frac{\partial^2 u}{\partial x^2} + b(x) \frac{\partial u}{\partial x}$$

$$u(0,x) = f(x).$$

(There is only one solution $u(t,x)$ of the PDE such that $u(t,\cdot)$ is in G for each $t \geq 0$.) Furthermore, $\langle u(t,\cdot),\, u(t,\cdot)\rangle \leq \langle f,f\rangle$ for each $t \geq 0$; *i.e.*, $\{T_t,\ t \geq 0\}$ is a contraction semigroup.

We can conclude, finally, that $\{T_t f\}$ has a limit in G as $t \to \infty$ and that the limit is constant. Let

$$p(t,x) = \frac{1}{a(x)} \exp\left(2 \int_0^x b(x)/a(s)\, ds\right) u(t,x)\ .$$

Then

$$\frac{\partial}{\partial t}\, p(t,x) = \frac{1}{a(x)} \exp\left(2 \int_0^x b(s)/a(s)\, ds\right) \frac{\partial}{\partial t}\, u(t,x),$$

$$\frac{\partial}{\partial x}\, (a(x)\, p(t,x)) = \frac{2b(x)}{a(x)} \exp\left(2 \int_0^x b(s)/a(s))\, ds\right) u(t,x)$$

$$+ \exp\left(2 \int_0^x b(s)/a(s)\, ds\right) \frac{\partial}{\partial x}\, u(t,x)$$

$$= 2\, b(x)\, p(t,x) + \exp\left(2 \int_0^x b(s)/a(s)\, ds\right) \frac{\partial}{\partial x}\, u(t,x),$$

and

$$\frac{\partial^2}{\partial x^2}\, (a(x)\, p(t,x)) = 2\, \frac{\partial}{\partial x}\, (b(x) p(t,x))$$

$$+ 2\, \frac{b(x)}{a(x)} \exp\left(2 \int_0^x b(s)/a(s)\, ds\right) \frac{\partial}{\partial x}\, u(t,x)$$

$$+ \exp\left(2 \int_0^x b(s)/a(s)\, ds\right) \frac{\partial^2}{\partial x^2}\, u(t,x).$$

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[17] ibid.

$$\frac{\partial}{\partial t} p(t,x) - \tfrac{1}{2} \frac{\partial^2}{\partial x^2} (a(x)p(t,x)) + \frac{\partial}{\partial x} (b(x)p(t,x))$$

$$= \frac{1}{a(x)} \exp \left(2 \int_0^x b(s)/a(s)\, ds\right) \frac{\partial}{\partial t} u(t,x) - \frac{\partial}{\partial x} (b(x)p(t,x))$$

$$- \frac{b(x)}{a(x)} \exp \left(2 \int_0^x b(s)/a(s)\, ds\right) \frac{\partial}{\partial x} u(t,x)$$

$$- \tfrac{1}{2} \exp \left(2 \int_0^x b(s)/a(s)\, ds\right) \frac{\partial^2}{\partial x^2} u(t,x) + \frac{\partial}{\partial x} (b(x)p(t,x))$$

$$= \frac{1}{a(x)} \exp \left(2 \int_0^x b(s)/a(s) ds\right) \left\{\frac{\partial}{\partial t} u(t,x) - \tfrac{1}{2} a(x) \frac{\partial^2}{\partial x^2} u(t,x)\right.$$

$$\left. - b(x) \frac{\partial}{\partial x} u(t,x)\right\} = 0.$$

Furthermore,

$$p(t,x) \to \frac{c}{a(x)} \exp \left(2 \int_0^x b(s)/a(s)\, ds\right)$$

as $t \to \infty$ for some constant c.

SECTION IV

SYSTEM IDENTIFICATION

DEVELOPMENT OF LINEAR EQUATIONS

For the method of moments in its simplest form, one tries to choose the parameters so the model moments approximate the empirical moments. For the method outlined here,[19] we establish some relations, linear in the parameters, which have model moments as coefficients. We then substitute the empirical moments in place of the model moments and solve in a least squares sense for the parameter estimates. In the case of boundaries, the relations probably will have to include some which are nonlinear.

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[19]ibid.

In this and the last section we will assume a two-dimensional model of the form

$$dX(t) = \begin{bmatrix} b_1(X(t)) \\ b_2(X(t)) \end{bmatrix} dt + \varepsilon\, dW(t).$$

The linear model becomes

$$dX(t) = \begin{bmatrix} b_{11}\, x + b_{12}\, y \\ b_{21}\, x + b_{22}\, y \end{bmatrix} dt + \varepsilon\, dW(t)\ .$$

Further, we will assume that there is a steady-state density p and

$$\lim_{x\to\pm\infty} x^i\, p(x,y) = 0 \qquad \text{and} \qquad \lim_{y\to\pm\infty} y^i\, p(x,y) = 0$$

uniformly on bounded sets for all i.

In general, we have (see Appendix A):

$$\tfrac{1}{2}\,\varepsilon^2 \frac{\partial^2 p}{\partial x^2} + \tfrac{1}{2}\,\varepsilon^2 \frac{\partial^2 p}{\partial y^2} - \frac{\partial(b_1 p)}{\partial x} - \frac{\partial(b_2 p)}{\partial y} = 0$$

which yields

$$0 = \tfrac{1}{2}\,\varepsilon^2 \int_{-\infty}^{x} \int_{-\infty}^{y} \frac{\partial^2 p}{\partial x^2}(u,v)\ dudv + \tfrac{1}{2}\,\varepsilon^2 \int_{-\infty}^{x} \int_{-\infty}^{y} \frac{\partial^2 p}{\partial y^2}(u,v)\ dudv$$

$$- \int_{-\infty}^{x} \int_{-\infty}^{y} \frac{\partial(b_1 p)}{\partial x}(u,v)\ dudv - \int_{-\infty}^{x} \int_{-\infty}^{y} \frac{\partial(b_2 p)}{\partial y}(u,v)\ dudv$$

$$= \tfrac{1}{2}\,\varepsilon^2 \int_{-\infty}^{y} \frac{\partial p}{\partial x}(x,v)\ dv + \tfrac{1}{2}\,\varepsilon^2 \int_{-\infty}^{x} \frac{\partial p}{\partial y}(u,y)\ du$$

$$- \int_{-\infty}^{y} b_1(x,v)\ p(x,v)\ dv - \int_{-\infty}^{x} b_2(u,y)\ p(u,y)\ du.$$

If we let y go to ∞ and integrate with respect to x from $-\infty$ to ∞, we obtain

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} b_1(x,y)\ \ p(x,y)\ dxdy = 0\ .$$

Interchanging the roles of x and y yields

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} b_2(x,y)\, p(x,y)\, dxdy = 0.$$

For i and j non-negative integers, let

$$m_{ij} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^i y^j\, p(x,y)\, dxdy.$$

In the linear case, the equations $m_{10}b_{i1} + m_{01}b_{i2} = 0$ contain no information since $m_{10} = m_{01} = 0$. However, we can repeat the above process, but first multiplying by either x or y; i.e.,

$$0 = \tfrac{1}{2}\,\varepsilon^2 \int_{-\infty}^{x} \int_{-\infty}^{y} \{u\, \frac{\partial^2 p}{\partial x^2}(u,v) + u\, \frac{\partial^2 p}{\partial y^2}(u,v)\}\, dudv$$

$$- \int_{-\infty}^{x} \int_{-\infty}^{y} \{u\, \frac{\partial(b_1 p)}{\partial x}(u,v) + u\, \frac{\partial(b_2 p)}{\partial y}(u,v)\}\, dudv$$

$$= \tfrac{1}{2}\,\varepsilon^2 \int_{-\infty}^{y} \{x\, \frac{\partial p}{\partial x}(x,v) - p(x,v)\}\, dv + \tfrac{1}{2}\,\varepsilon^2 \int_{-\infty}^{x} u\, \frac{\partial p}{\partial y}(u,y)\, du$$

$$- \int_{-\infty}^{y} \{x\, b_1(x,v)\, p(x,v) - \int_{-\infty}^{x} b_1(u,v)\, p(u,v)\, du\}\, dv$$

$$- \int_{-\infty}^{x} u\, b_2(u,y)\, s(u,v)\, du.$$

Let y go to ∞. Then

$$0 = \tfrac{1}{2}\,\varepsilon^2 \int_{-\infty}^{\infty} \{x\, \frac{\partial p}{\partial x}(x,y) - p(x,y)\}\, dy$$

$$- \int_{-\infty}^{\infty} \{x\, b_1(x,y)\, p(x,y) - \int_{-\infty}^{x} b_1(u,y)\, p(u,y)\, du\}\, dy.$$

Integrating with respect to x from -∞ to ∞ we have

$$0 = \tfrac{1}{2}\,\varepsilon^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{x\,\frac{\partial p}{\partial x}(x,y) - p(x,y)\}\,dxdy$$

$$- \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{x\, b_1(x,y)\, p(x,y) - \int_{-\infty}^{x} b_1(u,y)\, p(u,y)\, du\}\, dydx.$$

In the linear case this yields $m_{20}\, b_{11} + m_{11}\, b_{12} = -\tfrac{1}{2}\,\varepsilon^2$.

On the other hand, if we let x go to ∞ and integrate with respect to y from $-\infty$ to ∞ we obtain

$$0 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y\, b_1(x,y)\, p(x,y)\, dxdy + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, b_2(x,y)\, p(x,y)\, dxdy.$$

In the linear case, $m_{11}\, b_{11} + m_{02}\, b_{12} + m_{20}\, b_{21} + m_{11}\, b_{22} = 0$.

As before, interchanging the roles of x and y we also obtain the equation

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y\, b_2(x,y)\, p(x,y)\, dxdy = -\tfrac{1}{2}\,\varepsilon^2 .$$

We can either use higher moments to obtain enough equations or some other knowledge of the model. For instance, if the model is symmetric, i.e., $b_{12} = b_{21}$, we obtain the following system:

$$\begin{bmatrix} m_{20} & m_{11} & 0 & 0 \\ m_{11} & m_{02} & m_{20} & m_{11} \\ 0 & 0 & m_{11} & m_{02} \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{12} \\ b_{21} \\ b_{22} \end{bmatrix} = \begin{bmatrix} -\tfrac{1}{2}\,\varepsilon^2 \\ 0 \\ -\tfrac{1}{2}\,\varepsilon^2 \\ 0 \end{bmatrix}$$

<u>Example 1</u>

Suppose $\varepsilon = 1$ and the moments are given as $m_{10} = m_{01} = 0$, $m_{20} = \tfrac{1}{2}$, and $m_{11} = m_{02} = \tfrac{1}{4}$. The system above becomes

$$\begin{bmatrix} \tfrac{1}{2} & \tfrac{1}{4} & 0 & 0 \\ \tfrac{1}{4} & \tfrac{1}{4} & \tfrac{1}{2} & \tfrac{1}{4} \\ 0 & 0 & \tfrac{1}{4} & \tfrac{1}{4} \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{12} \\ b_{21} \\ b_{22} \end{bmatrix} = \begin{bmatrix} -\tfrac{1}{2} \\ 0 \\ -\tfrac{1}{2} \\ 0 \end{bmatrix}$$

which is seen to be satisfied by $b_{11} = -2$, $b_{12} = b_{21} = 2$, and $b_{22} = -4$. The model is

$$dX(t) = \begin{bmatrix} -2x + 2y \\ 2x - 4y \end{bmatrix} dt + dW(t).$$

EXAMPLES OF FLOWS WITH POTENTIALS

We cannot at present solve the equation

$$\tfrac{1}{2}\,\varepsilon^2 \frac{\partial^2 p}{\partial x^2} + \tfrac{1}{2}\,\varepsilon^2 \frac{\partial^2 p}{\partial y^2} - \frac{\partial(b_1 p)}{\partial x} - \frac{\partial(b_2 p)}{\partial y} = 0$$

for a non-negative summable function p. In the special case that $[b_1(x,y)\ b_2(x,y)]$ has a potential function ϕ, i.e., $\partial\phi/\partial x = b_1$ and $\partial\phi/\partial y = b_2$, then $p(x,y) = c\, e^{2\phi(x,y)}$, where

$$c = 1/\left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\,(2\phi(x,y))\, dxdy\right)$$

is the unique steady-state density. For this case, we can construct numerical examples avoiding simulations.

<u>Example 2</u>

Consider the nonlinear model

$$dX(t) = \begin{bmatrix} -4x^3 + 3x - y \\ -x - y \end{bmatrix} dt + \varepsilon\, dW(t).$$

The potential function is

$$\phi(x,y) = -x^4 + 3/2\, x^2 - xy - \tfrac{1}{2}\, y^2.$$

Clearly,

$$\lim_{x\to\pm\infty} x^i e^{2\phi(x,y)} = 0 \qquad \text{and} \qquad \lim_{y\to\pm\infty} y^i e^{2\phi(x,y)} = 0$$

uniformly on bounded sets for every i. Hence we can use the density $p(x,y) = e^{2\phi(x,y)}$ to generate some moments numerically and test the method outlined above. For this model, $m_{10} = m_{01} = m_{30} = m_{21} = m_{12} = m_{03} = 0$. So in order to have enough linear relations to determine the "unknown" parameters in the model

$$dX(t) = \begin{bmatrix} 4a_1x^3 + 2a_2x + a_3y \\ a_3x + 2a_4y \end{bmatrix} dt + \varepsilon\, dW(t),$$

we must use at least fourth order moments (i + j = 4).

Besides the equations generated before we add

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y^3 b_2(x,y)\, p(x,y)\, dxdy = -3/2\, \varepsilon^2\, m_{02} \quad .$$

These relations would give four equations. We can add two more by assuming that there are modes at (1, -1) and (-1, 1); i.e., $b_1(1, -1) = 0$ and $b_2(-1, 1) = 0$. Thus we have (assuming ε is known) the following system:

$$\begin{bmatrix} 4 & 2 & -1 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & m_{11} & 2m_{02} \\ 4m_{40} & 2m_{20} & m_{11} & 0 \\ & & (m_{20} + m_{02}) & \\ 4m_{31} & 2m_{11} & 2 & 2m_{11} \\ 0 & 0 & m_{13} & 2m_{04} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2}\varepsilon^2 \\ -\frac{1}{2}\varepsilon^2 \\ \\ 0 \\ -3/2\, \varepsilon^2\, m_{02} \end{bmatrix}$$

Since the coefficient matrix is known only approximately, we can increase the accuracy of our estimates by adding relations. As indicated above, the additional equations need not involve moments but may arise from some other knowledge we have of the model.

SECTION V

QUALITATIVE ANALYSIS

MULTIPLE MODES

The qualitative analysis of nonlinear stochastic systems is complicated by the possibility of steady-state distributions which are not Gaussian. Consider the one-dimensional model

$$dX(t) = -X(t)(X^2(t) - 1)\, dt + dWt.$$

The forward equation is

$$\frac{\partial p}{\partial t} = \frac{1}{2} \frac{\partial^2 p}{\partial x^2} + \frac{\partial}{\partial x} (x(x^2 - 1)p)$$

and steady-state is given by

$$\tfrac{1}{2}p'' + [x(x^2 - 1)p]' = 0.$$

Thus

$$p(x) = ce^{-2 \int_o^x s^3 - s \, ds} = me^{-x^4/2 + x^2} .$$

Note that $p'(x) = (-2x^3 + 2x)\, p(x) = 0$ has solutions $x = -1, 0, 1$. These critical points can be classified as: -1 and 1 are local maxima and 0 is a local minimum. Thus $p(x)$ is the density of a bimodal distribution.

The system must linger proportionately longer in neighborhoods of states -1 and 1 than in neighborhoods of 0. Thus the system might exhibit a kind of periodic motion visiting alternately neighborhoods of states -1 and 1 with some expected time. If the system were deterministic (eliminating the noise term), then the trajectory upon arriving in certain neighborhoods of either -1 or 1 would not exit. The long term behavior of Markov processes requires a descriptive apparatus substantially different from the stability analysis of deterministic systems.

FIRST PASSAGE TIMES

Consider the one-dimensional model $dX(t) = b(X(t)\, dt + \varepsilon\, dW(t))$. Suppose $x < \beta$. We want to compute the probability that $X(t) \geq B$ given $X(0) = x$.[20] In order to do this, we treat β as an absorbing boundary and so we must pause to discuss the corresponding boundary condition.[15 21]

Let $F(y,t|x) = \mathrm{Prob}(X(t) \leq y|X(0) = x)$ denote the conditional probability distribution function and $J(y,t|x)$ the rate at which probability is flowing in the positive y-direction at y at time t. Then $J = -\partial F/\partial t$ is called the probability flux and $\partial p/\partial t = \partial^2 F/\partial t \partial y = -\partial J/\partial y$; i.e.,

$$J(y,t|x) = b(y)\, p(y,t|x) - \tfrac{1}{2} a \frac{\partial}{\partial y} p(y,t\ x).$$

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[15]ibid.

[20]Matkowsky, B. J. and Schuss, Y., The Exit Problem for Randomly Perturbed Dynamical Systems,, SIAM J. Appl. Math., 33(1975), pp. 365-382.

[21]Friedman, A., Stochastic Differential Equations and Applications, Academic Press, 1975.

The concept of probability flux is useful in formulating the appropriate boundary condition.

With an absorbing barrier at β the Chapman-Kolmogorov equation becomes

$$p(z,t+s|x) = \int_{-\infty}^{\beta} p(z,t|y)\, p(y,s|x)\, dy.$$

As before, if $u(t,x)$ is any solution of the backward equation then

$$\int_{-\infty}^{\beta} u(0,z)\, p(z,t+\tau|x)\, dx = \int_{-\infty}^{\beta} u(t,y)\, p(y,\tau|x)\, dy.$$

One notes that differentiating the left-hand side wrt t or τ is equivalent. Hence, by virtue of the equality, differentiating the right-hand side by t or τ is also equivalent. Thus

$$\int_{-\infty}^{\beta} u(t,y) \frac{\partial}{\partial \tau} p(y,\tau|x)\, dy = \int_{-\infty}^{\beta} \frac{\partial}{\partial t} u(t,y)\, p(y,\tau|x)\, dy$$

(Substituting the expression for $\partial u/\partial t$ from the backwards equation into the right-hand expression, one has)

$$= \int_{-\infty}^{\beta} [\tfrac{1}{2} a \frac{\partial^2}{\partial y^2} u(t,y) + b(y) \frac{\partial}{\partial y} u(t,y)]\, p(y,\tau|x)\, dy \qquad (*)$$

(Integrating by parts leads to)

$$= \tfrac{1}{2} a(\frac{\partial}{\partial y} u(t,y))\, p(y,t|x) \Big|_{y=-\infty}^{y=\beta} + b(y)u(t,y)\, p(y,t|x) \Big|_{y=-\infty}^{y=\beta}$$

$$- \int_{-\infty}^{\beta} \{\tfrac{1}{2} a \frac{\partial}{\partial y} p(y,\tau|x) \frac{\partial}{\partial y} u(t,y) + \frac{\partial}{\partial y} (b(y)\, p(y,\tau|x))\, u(t,y)\}\, dy$$

(The probability density is zero at $-\infty$ and β; integrating the first term in the integrand by parts one has)

$$= -\tfrac{1}{2} a \frac{\partial}{\partial y} p(y,\tau|x)\, u(t,y) \Big|_{-\infty}^{\beta}$$

$$+ \int_{-\infty}^{\beta} \{\tfrac{1}{2} a \frac{\partial^2}{\partial y^2} p(y,\tau|x) - \frac{\partial}{\partial y} (b(y)\, p(y,\tau|x))\}\, u(t,y)\, dy$$

(Substituting from the forward equation the expression for $\partial p/\partial t$ into equation (*) and comparing with the expression directly above, one has)

or $J(\beta,\tau\ x)\ u(t,\beta) = 0$. Since in general $J(\beta,\tau|x) \neq 0$, then we must require that $u(t,\beta) = 0$, our boundary condition for the backward equation.

Thus the probability that $X(t) \leq \beta$ given $X(0) = x$ is found by solving

$$\frac{\partial u}{\partial t} = \tfrac{1}{2}\, a \frac{\partial^2 u}{\partial y^2} + b(y) \frac{\partial u}{\partial y}$$

$$u(0,y) = \begin{cases} 1 & \text{for } y \leq \beta \\ 0 & \text{for } y > \beta \end{cases}$$

$$u(t,\beta) = 0$$

Further, $1 - u(t,x)$ is the proportion of trajectories that reach β in $[0,t]$ and

$$-\frac{\partial}{\partial t} u(t,x) = J(\beta,t|x).$$

Let $w(t,x) = -\partial/\partial\tau\ u(t,x)$. Then differentiating the backward equation yields

$$\frac{\partial w}{\partial t} = \tfrac{1}{2}\, a \frac{\partial^2 w}{\partial x^2} + b(x) \frac{\partial w}{\partial x} \quad ,$$

$$w(t,\beta) = \delta(t),$$

$$w(t,-\infty) = 0,$$

and

$$w(0,x) = \delta(\beta - x)$$

and $\int_0^\infty w(t,x)\,dt$ is the proportion of trajectories which leave $(-\infty,\beta)$ eventually.

Let

$$T = \inf\ \{t:\ X(t) = \beta,\ X(0) = x\}$$

the exit of first passage time and

$$z(x) = E(T) = \int_0^\infty t w(t,x)\,dt.$$

If Prob $(T < \infty) = 1$ or, equivalently

$$\int_0^\infty w(t,x)\,dt = 1,$$

then

$$\int_o^\infty t \frac{\partial}{\partial t} w(t,x)\, dt = \tfrac{1}{2}\, a \frac{\partial^2}{\partial x^2} \int_0^\infty tw(t,x)\, dt + b(x) \frac{\partial}{\partial x} \int_0^\infty tw(t,x)\, dt$$

$$= t\, w(t,x) \Big|_o^\infty - \int_o^\infty w(t,x)\, dt$$

or

$$\tfrac{1}{2}\, a\, z'' + b(x)\, z' = -1,\ z(\beta) = 0, \text{ and } z'\ (-\infty) = 0.$$

Examples

We want to compare the first passage times from 0 to 2 for the following models:

1. $dX(t) = -2\, X(t)\, dt + \varepsilon\, dW(t)$
2. $dX(t) = -(X^3(t) - 3\, X^2(t) + 2X(t))\, dt + \varepsilon\, dW(t)$
3. $dX(t) = -(2X^3(t) - 5\, X^2(t) + 2X(t))\, dt + \varepsilon\, dW(t)$
4. $dX(t) = -(4X^3(t) - 9\, X^2(t) + 2X(t))\, dt + \varepsilon\, dW(t)$.

Note that $b(x)$ has zeros at 0 and 2 for each of 2 through 4 and a zero between 0 and 2 which is nearer to 0 for 4 than for 3 and nearer for 3 than for 2. The expected first passage times for the four models are approximately 500, 4, 2, and 1.

In all four models, the tendency for a trajectory in a small enough neighborhood of 0 is to return to 0. For the last three models, a trajectory to the right of the middle zero of $b(x)$ has a tendency to go to 2. The closer that zero is to 0 the easier for the random disturbance to push the systm into the region of attraction of 2.

SECTION VI

CONTROL PROBLEMS

FEEDBACK CONTROL

The first objective, using an analogy with linear systems, is to control the given system so the steady-state distribution is unimodal with mode at 0. A second objective might be to fix moments higher than the first.

Consider a one-dimensional model $dX(t) = b(X(t))\ dt + \varepsilon\ dW(t)$. If the model is linear, i.e., $b(x) = bx$, then the question of stabilizing the system is trivial. Looking at the deterministic system $dX(t) = (b \cdot x(t) + u(t))\ dt$ we can choose $u(t) = c \cdot x(t)$ where $b + c < 0$. In the nonlinear case if we require $u(t)$ to be of the form $u(t) = c \cdot x(t)$, there might be no such $u(t)$ which stabilizes the system. However, even if we require that

$$dX(t) = b(X(t))\ dt + \varepsilon\ dW(t)$$

has a steady-state density we can wonder if there is a number co so that the steady-state density of

$$dX(t) = (b(X(t)) + c\ X(t)\ dt + \varepsilon\ dW(t)$$

is unimodal with mode at 0.

The following examples illustrate the answer to this question.

Example 1. The scaler nonlinear system

$$dX(t) = [-X(t)\ (X^2(t) - 1) + c\ X(t)]\ dt + dW(t)$$

has steady-state density

$$p(x) = m \exp(-x \cdot 4/2 + (1 + c)\ x^2).$$

Further,

$$p'(x) = p(x) \cdot (-2x^3 + 2\ (1 + c)\ x)$$

has exactly one zero if $c \leq -1$. Clearly, this generalizes if $b(x)$ can be differentiated.

Suppose the two-dimensional model

$$dX(t) = \begin{bmatrix} b_1(X(t)) \\ b_2(X(t)) \end{bmatrix} dt + \varepsilon\ dW(t)$$

has a potential and $b_1(0,0) = b_2(0,0) = 0$. Sufficient conditions on a 2x2 matrix C so

$$dX(t) = \begin{bmatrix} b_1(X(t)) + c_{11}X_1(t) + c_{12}X_2\ (t) \\ b_2(X(t)) + c_{21}X_1(t) + c_{22}X_2(t) \end{bmatrix} dt + \varepsilon\ dW(t)$$

has a mode at (0,0) are

1. $c_{12} = c_{21}$

2. $c_{11} < -\frac{\partial b_2}{\partial x}(0,0)$

3. $c_{22} < -\frac{\partial b_2}{\partial y}(0,0)$

4. $[\frac{\partial b_1}{\partial x}(0,0) + c_{11}]\,[\frac{\partial b_2}{\partial y}(0,0) + c_{22}] - [\frac{\partial b_1}{\partial y}(0,0) + c_{12}]^2 > 0.$

Example 2. Consider the system

$$dX(t) = \begin{bmatrix} -4X_1^3(t) + 3X_1(t) - X_2(t) \\ -X_1(t) - X_2(t) \end{bmatrix} dt + \varepsilon\, dW(t)$$

whose steady-state density has local maxima at (1, -1) and (-1, 1). The indicated partials are $\partial b_1/\partial x = -12x^2 + 3$, $\partial b_1/\partial y = -1$, $\partial b_2/\partial x = -1$, and $\partial b_2/\partial y = -1$. Hence sufficient conditions are $c_{11} < -3$, $c_{22} < 1$, $c_{12} = c_{21}$, and $(3 + c_{11})(-1 + c_{22}) - (-1 + c_{12})^2 > 0$. If we choose $c_{12} = c_{21} = 0$, $c_{11} = -4$, and $c_{22} = 0$ then $b_1 = -4x^3 - x$ and $b_2 = -y$. For these choices, the controlled system

$$dX(t) = \begin{bmatrix} -4X_1^3(t) - X_1(t) \\ -X^2(t) \end{bmatrix} dt + \varepsilon\, dW(t)$$

has unimodal steady-state distribution with mode at (0,0). The question of whether this result can always be obtained is open.

No optimal control problem has been formulated.[22] [23] The formulation of a significant optimization problem must be preceded by the formulation of reasonable objectives. This in turn requires more insight into the qualitative analysis for nonlinear stochastic systems.

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[22]Fleming, W. H. and Rishel, R. W., Deterministic and Stochastic Control, Springer-Verlag, 1975.

[23]Kushner, H. J., Stochastic Stability and Control, Academic Press, 1967.

APPENDIX A

THE LINEAR RELATIONS FOR THE PARAMETERS IN TERMS OF THE MOMENTS

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THE LINEAR RELATIONS FOR THE PARAMETERS IN TERMS OF THE MOMENTS

The starting point for obtaining linear relations for the parameters of the n-dimensional system is the steady-state forward equation.

$$0 = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 p}{\partial x_i dx_j} - \sum_{j=1}^{n} \frac{\partial(b_j p)}{\partial x_j} .$$

Multiplying by $(x_k)^m$ and integrating x_i from $-\infty$ to ∞ for $i = 1,2,\ldots,n$, we obtain

$$0 = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} (u_k)^m \frac{\partial^2 p}{\partial x_i \partial x_j} (u)du$$

$$- \sum_{j=1}^{m} \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} (u_k)^m \frac{\partial(b_j p)}{\partial x_j} (u)du$$

Let $x_i \to \infty (i \neq k)$ and integrate from $-\infty$ to ∞ with respect to x_k. Then

$$0 = \int_{R^n} \int_{-\infty}^{x_k} (u_k^m) \left[\frac{1}{2} a_{kk} \frac{\partial^2 p}{\partial x_k^2} - \frac{\partial(b_k p)}{\partial x_k}\right] (x_1, \ldots, u_k, \ldots, u_n)du_k dx$$

$$= \frac{a_{kk}}{2} \left\{ \int_{R^n} (x_k)^m \frac{\partial p}{\partial x_k}(x)dx - m \int_{R^n} \int_{-\infty}^{x_k} (u_k)^{m-1} \frac{\partial p}{\partial x_k} (x_1, \ldots, u_k, \ldots, x_n)du_k dx\right\}$$

$$- \int_{R^n}(x_k)^m b_k(x) p(x)dx + m \int_{R^n} \int_{-\infty}^{x_k} (u_k)^{m-1} [b_k p](x_1, \ldots, u_k, \ldots, x_n)du_k dx$$

$$= \frac{a_{kk}}{2} \left\{-m \int_{R^n} (x_k)^{m-1} p(x)dx + m \int_{R^n} (x_k)^m \frac{\partial p}{\partial x_k}(x)dx\right\}$$

$$- (1 + m) \int_{R^n} (x_k)^m b_k(x)p(x)dx$$

$$= - \frac{m(m + 1)}{2} a_{kk} \int_{R^n} (x_k)^{m-1} p(x)dx - (1 + m) \int_{R^n}(x_k)^m b_k(x)p(x)dx.$$

Or

$$\int_{R^n} (x_k)^m b_k(x)p(x)dx = -\frac{m}{2} a_{kk} \int_{R^n} (x_k)^{m-1} p(x)dx.$$

On the other hand, for $i \neq k$ let $x_j \to \infty$ $(j \neq i)$ and integrate from $-\infty$ to ∞ with respect to x_i.

$$0 = a_{ik} \int_{R^n} \int_{-\infty}^{x_i} (x_k)^m \frac{\partial^2 p}{\partial x_i \partial x_k} (x_1, \ldots u_i, \ldots x_n) du_i dx$$

$$+ \tfrac{1}{2} a_{kk} \int_{R^n} \int_{-\infty}^{x_i} (x_k)^m \frac{\partial^2 p}{\partial x_i^2} (x_1, \ldots, u_i, \ldots, x_n) du_i dx$$

$$+ \tfrac{1}{2} a_{ii} \int_{R^n} \int_{-\infty}^{x_i} (x_k)^m \frac{\partial^2 p}{\partial x_i^2} (x_1, \ldots, u_i, \ldots, x_n) du_i dx$$

$$- \int_{R^n} \int_{-\infty}^{x_i} (x_k)^m \frac{\partial (b_i p)}{\partial x_i} (x_1, \ldots, u_i, \ldots, x_n) du_i dx$$

$$\int_{R^n} \int_{-\infty}^{x_i} (x_k)^m \frac{\partial (b_i p)}{\partial x_k} (x_1, \ldots, u_i, \ldots, x_n)\, du_i dx$$

$$= -m\, a_{ik} \int_{R^n} (x_k)^{m-1} p(x)dx - \frac{m}{2} a_{kk} \int_{R^n} \int_{-\infty}^{x_i} \frac{\partial p}{\partial x_k} (x_1, \ldots, u_i, \ldots, x_n) du_i dx$$

$$+ \tfrac{1}{2} a_{ii} \int_{R^n} (x_k)^m \frac{\partial p}{\partial x_i}(x)dx - \int_{R^n} (x_k)^m b_i(x)p(x)dx$$

$$+m \int_{R^n} \int_{-\infty}^{x_i} (x_k)^{m-1} [b_k p] (x_1, \ldots, u_i, \ldots, x_n) du_i dx.$$

If $m = 1$, then

$$0 = a_{ik} + \int_{R^n} x_k\, b_i(x)p(x)dx + \int_{R^n} x_i b_k(x)p(x)dx.$$

If $m > 1$, then

$$0 = -\, m a_{ik} \int_{R^n} (x_k)^{m-1}\, p(x)dx + \frac{m(m-1)}{2}\, a_{kk} \int_{R^n} \int_{-\infty}^{x_i} (x_k)^{m-2}\, p(x_1,\ldots,u_i,\ldots,x_n)du_i dx$$

$$-\int_{R^n} (x_k)^m\, b_i(x)p(x)dx - m \int_{R^n} (x_k)^{m-1}\, x_i b_k(x)p(x)dx.$$

Or

$$\int_{R^n} (x_k)^m\, b_i(x)p(x)dx + m \int_{R^n} (x_k)^{m-1}\, x_i b_k(x)p(x)dx$$

$$= \quad -m\, a_{ik} \int_{R^n} (x_k)^{m-1}\, p(x)dx - \frac{m(m-1)}{2}\, a_{kk} \int_{R^n} (x_k)^{m-2}\, x_i p(x)dx.$$

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